Lemma 12.1 : Let $n \in \mathbb{N}$, $n \geq 2$ be arbitrary. Then,

$$\int \text{Cos}^n\,\theta\, d\theta = \frac{\text{Cos}^{n-1}\,\theta \cdot \text{Sin}\,\theta}{n} + \frac{n-1}{n} \int \text{Cos}^{n-2}\, d\theta \qquad [1].$$

Proof :

This is an integral reduction formula, taken from Calculus and Analytic Geometry, 8th Edition, by Thomas / Finney. The reduction formula is given in the book, but no proof is provided. We will provide the proof of this reduction formula here. Consider the *R.H.S.* of equation [1] above. Taking the derivative, with respect to θ, we have

$$\frac{d}{d\theta}\left(\frac{\text{Cos}^{n-1} \cdot \text{Sin}\,\theta}{n} + \frac{n-1}{n} \int \text{Cos}^{n-2}\,\theta\, d\theta\right) = \frac{-(n-1)\,\text{Cos}^{n-2}\,\theta \cdot \text{Sin}^2\,\theta + \text{Cos}^n\,\theta}{n} + \frac{n-1}{n}\,\text{Cos}^{n-2}\,\theta,$$

where the result for the first term of the summand is obtained using the product rule for derivatives, and the second term is obtained by the application of the Fundamental Theorem of Integral Calculus. Hence,

$$\frac{-(n-1)\,\text{Cos}^{n-2}\,\theta \cdot \text{Sin}^2\,\theta + \text{Cos}^n\,\theta}{n} + \frac{n-1}{n}\,\text{Cos}^{n-2}\,\theta = \frac{(n-1)-(n-1)}{n}\,\text{Cos}^{n-2}\,\theta + \frac{n-1+1}{n}\,\text{Cos}^n\,\theta$$
$$= \text{Cos}^n\,\theta,$$

where we obtained this result using the relationship $\text{Cos}^2\,\theta + \text{Sin}^2\,\theta = 1$. Taking the derivative of the *L.H.S.* of equation [1] above, with respect to θ, we have

$$\text{Cos}^n\,\theta = \frac{d}{d\theta} \int \text{Cos}^n\,\theta\, d\theta = \frac{d}{d\theta}\left(\frac{\text{Cos}^{n-1}\,\theta \cdot \text{Sin}\,\theta}{n} + \frac{n-1}{n} \int \text{Cos}^{n-2}\, d\theta\right).$$

Thus, integrating with respect to θ, we have

$$\int \text{Cos}^n\,\theta\, d\theta = \frac{\text{Cos}^{n-1}\,\theta \cdot \text{Sin}\,\theta}{n} + \frac{n-1}{n} \int \text{Cos}^{n-2}\, d\theta,$$

which establishes the result of the Lemma.□

Lemma 12.2 : Let $k \in \mathbb{N}$ be arbitrary. Then,

$$\int \mathrm{Cos}^{2k}\theta\, d\theta = P + Q + C,$$

where

$$P = \sum_{i=1}^{k} \frac{\mathrm{Cos}^{2(k-i)+1}\theta \cdot \mathrm{Sin}\,\theta \prod_{j=0}^{i-1}(2(k-j)+1)}{(2k+1)\prod_{j=0}^{i-1}(2(k-j))},$$

$$Q = \frac{\prod_{i=1}^{k}(2(k-i)+1)}{\prod_{i=0}^{k-1}(2(k-i))} \cdot \theta, \text{ and}$$

$C \in \mathbb{R}$ is *a* constant.

Proof :

The proof is by mathematical induction, with respect to k. To establish the basis for induction, suppose $k = 1$. Then, applying the result from Lemma 12.1, we have

$$\int \mathrm{Cos}^{2k}\theta\, d\theta = \int \mathrm{Cos}^{2}\theta\, d\theta$$

$$= \frac{\mathrm{Cos}\,\theta \cdot \mathrm{Sin}\,\theta}{2} + \frac{1}{2}\int \mathrm{Cos}^{0}\theta\, d\theta$$

$$= \frac{\mathrm{Cos}\,\theta \cdot \mathrm{Sin}\,\theta}{2} + \frac{\theta}{2} + C$$

$$= P + Q + C,$$

which established the basis for induction. Now, suppose the result holds for all $i \leq k$. We need to show that the result holds for $i = k+1$. So, applying the reduction formula from Lemma 12.1, we have

$$\int \mathrm{Cos}^{2(k+1)}\theta\, d\theta = \frac{\mathrm{Cos}^{2k+1}\theta \cdot \mathrm{Sin}\,\theta}{2(k+1)} + \frac{2k+1}{2(k+1)}\int \mathrm{Cos}^{2k}\, d\theta$$

$$= \frac{\mathrm{Cos}^{2k+1}\theta \cdot \mathrm{Sin}\,\theta}{2(k+1)} + \frac{2k+1}{2(k+1)}(P + Q + C).$$

Now,

$$\frac{2k+1}{2(k+1)} P = \sum_{i=1}^{k} \frac{\mathrm{Cos}^{2(k-i)+1}\theta \cdot \mathrm{Sin}\,\theta \prod_{j=0}^{i-1}(2(k-j)+1)}{2(k+1)\prod_{j=0}^{i-1}(2(k-j))},$$

and

$$\frac{2k+1}{2(k+1)} Q = \frac{\prod_{i=1}^{k+1}(2(k+1-i)+1)}{\prod_{i=0}^{k}(2(k+1-m))} \cdot \theta.$$

Also,

$$\frac{\mathrm{Cos}^{2k+1}\theta \cdot \mathrm{Sin}\,\theta}{2(k+1)} + \frac{2k+1}{2(k+1)} P = \sum_{i=1}^{k+1} \frac{\mathrm{Cos}^{2(k+1-i)+1}\theta \cdot \mathrm{Sin}\,\theta \prod_{j=0}^{i-1}(2(k+1-j)+1)}{(2(k+1)+1)\prod_{j=0}^{i-1}(2(k+1-j))},$$

so that

$$\int \text{Cos}^{2(k+1)}\theta\, d\theta = \sum_{i=1}^{k+1} \frac{\text{Cos}^{2(k+1-i)+1}\theta \cdot \text{Sin}\,\theta \prod_{j=0}^{i-1}(2(k+1-j)+1)}{(2(k+1)+1)\prod_{j=0}^{i-1}(2(k+1-j))} + \frac{\prod_{i=1}^{k+1}(2(k+1-i)+1)}{\prod_{i=0}^{k}(2(k+1-m))} \cdot \theta + C_1,$$

where $C_1 = \frac{2k+1}{2(k+1)}$ C is *a* constant. This establishes the induction step. Therefore, by mathematical induction

$$\int \text{Cos}^{2k}\theta\, d\theta = P + Q + C,$$

for all $k \in \mathbb{N}$, where

$$P = \sum_{i=1}^{k} \frac{\text{Cos}^{2(k-i)+1}\theta \cdot \text{Sin}\,\theta \prod_{j=0}^{i-1}(2(k-j)+1)}{(2k+1)\prod_{j=0}^{i-1}(2(k-j))},$$

$$Q = \frac{\prod_{i=1}^{k}(2(k-i)+1)}{\prod_{i=0}^{k-1}(2(k-i))} \cdot \theta, \text{ and}$$

$C \in \mathbb{R}$ is *a* constant.□

Lemma 12.3 : Let $a, b \in \mathbb{N}$ for which $a = 2k_1 - 1$ and $b = 2k_2 - 1$, some $k_1, k_2 \in \mathbb{N}$. Then,

$$(1)\ \int_0^t x^{\frac{a}{2}} (1-x)^{\frac{b}{2}}\, dx = 2\int_0^{\text{Sin}^{-1}\sqrt{t}} \text{Sin}^{a+1}\theta \cdot \text{Cos}^{b+1}\theta\, d\theta,$$

where $0 < t < 1$, and

$$(2)\ \int_0^{\text{Sin}^{-1}\sqrt{t}} \text{Sin}^{a+1}\theta \cdot \text{Cos}^{b+1}\theta\, d\theta = \sum_{i=0}^{\frac{a+1}{2}} \binom{\frac{a+1}{2}}{i} (-1)^{\frac{a+1}{2}-i} \left(P_{\frac{a+b}{2}+1-i} + Q_{\frac{a+b}{2}+1-i}\right),$$

where

$$P_k = \sum_{i=1}^{k} \frac{\text{Cos}^{2(k-i)+1}\theta \cdot \text{Sin}\,\theta \prod_{j=0}^{i-1}(2(k-j)+1)}{(2k+1)\prod_{j=0}^{i-1}(2(k-j))}$$

and

$$Q_k = \frac{\prod_{i=1}^{k}(2(k-i)+1)}{\prod_{i=0}^{k-1}(2(k-i))} \cdot \theta,$$

$k = \frac{a+b}{2} + 1 - i$, and $\theta = \text{Sin}^{-1}\sqrt{t}$. □

Proof :

(1) We first only consider the integral $\int_0^t x^{\frac{a}{2}} (1-x)^{\frac{b}{2}}\, dx$. Let $u = \sqrt{x}$, so that $du = \frac{1}{2\sqrt{x}}\, dx$. Now,

$$\begin{aligned} x^{\frac{a}{2}} \cdot dx &= \frac{x^{\frac{a}{2}+\frac{1}{2}}}{\sqrt{x}} \cdot dx \\ &= 2\,(u^2)^{\frac{a+1}{2}} \cdot du \\ &= 2\,u^{a+1} \cdot du. \end{aligned}$$

Thus,

$$\int_0^t x^{\frac{a}{2}} (1-x)^{\frac{b}{2}}\, dx = \int_0^{\sqrt{t}} 2\,u^{a+1} (1-u^2)^{\frac{b}{2}}\, du.$$

Now, let $u = \text{Sin}\,\theta$, so that $du = \text{Cos}\,\theta\, d\theta$. Here, $0 < u < 1$, so that $0 < \text{Sin}\,\theta < 1$. Hence,

$$0 = \text{Sin}^{-1}(0) < \theta < \text{Sin}^{-1}(1) = \frac{\pi}{2},$$

so that

$$\begin{aligned} \int_0^{\sqrt{t}} 2\,u^{a+1}(1-u^2)^{\frac{b}{2}}\, du &= 2\int_0^{\text{Sin}^{-1}\sqrt{t}} \text{Sin}^{a+1}\theta\,(1-\text{Sin}^2\theta)^{\frac{b}{2}} \cdot \text{Cos}\,\theta\, d\theta \\ &= 2\int_0^{\text{Sin}^{-1}\sqrt{t}} \text{Sin}^{a+1}\theta\,(\text{Cos}^2\theta)^{\frac{b}{2}} \cdot \text{Cos}\,\theta\, d\theta \\ &= 2\int_0^{\text{Sin}^{-1}\sqrt{t}} \text{Sin}^{a+1}\theta \cdot \text{Cos}^{b}\theta \cdot \text{Cos}\,\theta\, d\theta \end{aligned}$$

$$= 2 \int_0^{\text{Sin}^{-1}\sqrt{t}} \text{Sin}^{a+1}\,\theta \cdot \text{Cos}^{b+1}\,\theta\, d\theta.$$

Thus, $\int_0^t x^{\frac{a}{2}} (1-x)^{\frac{b}{2}}\, dx = 2 \int_0^{\text{Sin}^{-1}\sqrt{t}} \text{Sin}^{a+1}\,\theta \cdot \text{Cos}^{b+1}\,\theta\, d\theta$, as desired.

(2) Here,

$$\int_0^{\text{Sin}^{-1}\sqrt{t}} \text{Sin}^{a+1}\,\theta \cdot \text{Cos}^{b+1}\,\theta\, d\theta = \int_0^{\text{Sin}^{-1}\sqrt{t}} (1-\text{Cos}^2\,\theta)^{\frac{a+1}{2}} \cdot \text{Cos}^{b+1}\,\theta\, d\theta.$$

Since $a = 2k_1 - 1$, some $k_1 \in \mathbb{N}$, then $a+1 = 2k_1$. This implies that $\frac{a+1}{2} = k_1 \in \mathbb{N}$. Hence,

we proceed to expand the term $(1-\text{Cos}^2\,\theta)^{\frac{a+1}{2}}$. So,

$$(1-\text{Cos}^2\,\theta)^{\frac{a+1}{2}} = \sum_{i=0}^{\frac{a+1}{2}} \binom{\frac{a+1}{2}}{i} (1)^i (-\text{Cos}^2\,\theta)^{\frac{a+1}{2}-i}$$

$$= \sum_{i=0}^{\frac{a+1}{2}} \binom{\frac{a+1}{2}}{i} (-1)^{\frac{a+1}{2}-i} (\text{Cos}^2\,\theta)^{\frac{a+1}{2}-i}$$

$$= \sum_{i=0}^{\frac{a+1}{2}} \binom{\frac{a+1}{2}}{i} (-1)^{\frac{a+1}{2}-i} (\text{Cos}\,\theta)^{a+1-2i}.$$

Thus,

$$\int_0^{\text{Sin}^{-1}\sqrt{t}} (1-\text{Cos}^2\,\theta)^{\frac{a+1}{2}} \cdot \text{Cos}^{b+1}\,\theta\, d\theta = \int_0^{\text{Sin}^{-1}\sqrt{t}} \sum_{i=0}^{\frac{a+1}{2}} \binom{\frac{a+1}{2}}{i} (-1)^{\frac{a+1}{2}-i} (\text{Cos}\,\theta)^{a+1-2i} \cdot \text{Cos}^{b+1}\,\theta\, d\theta$$

$$= \sum_{i=0}^{\frac{a+1}{2}} \binom{\frac{a+1}{2}}{i} (-1)^{\frac{a+1}{2}-i} \int_0^{\text{Sin}^{-1}\sqrt{t}} \text{Cos}^{a+b+2-2i}\,\theta\, d\theta.$$

Recall, in Lemma 12.2, we showed that $\int \text{Cos}^{2k}\,\theta\, d\theta = P_k + Q_k + C$,

where

$$P_k = \sum_{i=1}^{k} \frac{\text{Cos}^{2(k-i)+1}\,\theta \cdot \text{Sin}\,\theta \prod_{j=0}^{i-1} (2(k-j)+1)}{(2k+1)\prod_{j=0}^{i-1} (2(k-j))},$$

$$Q_k = \frac{\prod_{i=1}^{k} (2(k-i)+1)}{\prod_{i=0}^{k-1} (2(k-i))} \cdot \theta, \text{ and}$$

$C \in \mathbb{R}$ is a constant. Then, we have

$$\sum_{i=0}^{\frac{a+1}{2}} \binom{\frac{a+1}{2}}{i} (-1)^{\frac{a+1}{2}-i} \int_0^{\text{Sin}^{-1}\sqrt{t}} \text{Cos}^{a+b+2-2i}\,\theta\, d\theta = \sum_{i=0}^{\frac{a+1}{2}} \binom{\frac{a+1}{2}}{i} (-1)^{\frac{a+1}{2}-i} \left(P_{\frac{a+b}{2}+1-i} + Q_{\frac{a+b}{2}+1-i}\right) \Bigg|_{\theta=0}^{\theta=\text{Sin}^{-1}\sqrt{t}}.$$

The application of the lower limit of integration, $\theta = 0$, to each of the terms $P_{\frac{a+b}{2}+1-i} + Q_{\frac{a+b}{2}+1-i}$, yields zero.

The application of our upper limit of integration, $\theta = \text{Sin}^{-1}\sqrt{t}$, yields the desired result, specified by the Lemma.

Thus, $\int_0^{\text{Sin}^{-1}\sqrt{t}} \text{Sin}^{a+1}\theta \cdot \text{Cos}^{b+1}\theta \, d\theta = \sum_{i=0}^{\frac{a+1}{2}} \binom{\frac{a+1}{2}}{i} (-1)^{\frac{a+1}{2}-i} \left(P_{\frac{a+b}{2}+1-i} + Q_{\frac{a+b}{2}+1-i}\right)$,

where

$$P_k = \sum_{i=1}^{k} \frac{\text{Cos}^{2(k-i)+1}\theta \cdot \text{Sin}\,\theta \prod_{j=0}^{i-1}(2(k-j)+1)}{(2k+1)\prod_{j=0}^{i-1}(2(k-j))}$$

and

$$Q_k = \frac{\prod_{i=1}^{k}(2(k-i)+1)}{\prod_{i=0}^{k-1}(2(k-i))} \cdot \theta,$$

$k = \frac{a+b}{2} + 1 - i$, and $\theta = \text{Sin}^{-1}\sqrt{t}$. □

<u>Lemma 12.5 :</u> Let $c \in \mathbb{N}^*$ be arbitrary, where $\mathbb{N}^* = \mathbb{N} \bigcup \{0\}$. Then,

$$\frac{\prod_{j=1}^{c+1} (2\,(c+1-j)+1)}{\prod_{j=0}^{c} 2\,(c+1-j)} = \frac{(-1)^{c+1}\sqrt{\pi}}{(c+1)!\,\Gamma\left(-c-\frac{1}{2}\right)}.$$

Proof :

The proof is by mathematical induction, with respect to c. To establish the basis for induction, suppose that $c = 0$.

Then, $\frac{\prod_{j=1}^{c+1} (2\,(c+1-j)+1)}{\prod_{j=0}^{c} 2\,(c+1-j)} = \frac{1}{2}$. Also,

$$\frac{(-1)^{c}\sqrt{\pi}}{(c+1)!\,\Gamma\left(-c-\frac{1}{2}\right)} = \frac{(-1)^{1}\sqrt{\pi}}{1!\cdot\Gamma\left(-\frac{1}{2}\right)}$$

$$= \frac{-\sqrt{\pi}}{-2\sqrt{\pi}}$$

$$= \frac{1}{2}$$

$$= \frac{\prod_{j=1}^{c+1} (2\,(c+1-j)+1)}{\prod_{j=0}^{c} 2\,(c+1-j)},$$

so that the basis for induction holds. Now let $c \in \mathbb{N}$ be arbitrary. Suppose the result holds for all $k \in \mathbb{N}$, $k \leq c$. We need to show that the result holds for $k = c+1$. Here,

$$\frac{\prod_{j=1}^{c+2} (2\,(c+2-j)+1)}{\prod_{j=0}^{c+1} 2\,(c+2-j)} = \frac{(2\,(c+1)+1)\prod_{j=2}^{c+2} (2\,(c+2-j)+1)}{(2\,(c+2))\prod_{j=1}^{c+1} 2\,(c+2-j)}$$

$$= \frac{(2\,(c+1)+1)\,(-1)^{c+1}\sqrt{\pi}}{(2\,(c+2))\,(c+1)!\,\Gamma\left(-c-\frac{1}{2}\right)}.$$

Now,

$$\Gamma\left(-c-\frac{1}{2}\right) = \frac{(-2)^{c+1}\sqrt{\pi}}{\prod_{i=1}^{c} (2\,i+1)}$$

$$= \frac{(-1)^{c+1}\,(2)^{c+1}\sqrt{\pi}}{\prod_{i=1}^{c} (2\,i+1)}$$

$$= (-1)^{c+1}\,\frac{2^{c+1}\sqrt{\pi}}{\prod_{i=1}^{c} (2\,i+1)}.$$

Thus,

$$\frac{(2\,(c+1)+1)\,(-1)^{c+1}\sqrt{\pi}}{(2\,(c+2))\,(c+1)!\,\Gamma\left(-c-\frac{1}{2}\right)} = \frac{(2\,(c+1)+1)\,(-1)^{c+1}\sqrt{\pi}\,\left(\prod_{i=1}^{c} (2\,i+1)\right)}{(2\,(c+2))\,(c+1)!\,\left(2^{c+1}\sqrt{\pi}\,(-1)^{c+1}\right)}$$

$$= \frac{(-1)^{c+1}\sqrt{\pi}\,\prod_{i=1}^{c+1} (2\,i+1)}{(c+2)!\,2^{c+2}\sqrt{\pi}\,(-1)^{c+1}}$$

$$= \frac{(-1)^{c+2}\sqrt{\pi}\prod_{i=1}^{c+1}(2i+1)}{(c+2)!\,2^{c+2}\sqrt{\pi}\,(-1)^{c+2}}$$

$$= \frac{(-1)^{c+2}\sqrt{\pi}}{(c+2)!\,\frac{(-1)^{c+2}2^{c+2}\sqrt{\pi}}{\prod_{i=1}^{c+1}(2i+1)}}$$

$$= \frac{(-1)^{c+2}\sqrt{\pi}}{(c+2)!\,\Gamma\left(-(c+1)-\frac{1}{2}\right)},$$

so that

$$\frac{\prod_{j=1}^{c+2}(2(c+2-j)+1)}{\prod_{j=0}^{c+1}2(c+2-j)} = \frac{(-1)^{c+2}\sqrt{\pi}}{(c+2)!\,\Gamma\left(-(c+1)-\frac{1}{2}\right)},$$

as desired. Therefore, by mathematical induction $\frac{\prod_{j=1}^{c+1}(2(c+1-j)+1)}{\prod_{j=0}^{c}2(c+1-j)} = \frac{(-1)^{c+1}\sqrt{\pi}}{(c+1)!\,\Gamma\left(-c-\frac{1}{2}\right)}$, for all $c \in \mathbb{N}^*$. □

Proposition 12.1 : Let $n, m \in \mathbb{N}$ be arbitrary and let $X \sim F_{m,n}$. Let $c = \frac{m}{2}$, let $d = \frac{n}{2}$, and let $Y \sim \text{Beta}(c, d)$.

Then,

$$P(X \leq x) = P\left(Y \leq \frac{m \cdot x}{n + m \cdot x}\right). \square$$

Proof :

We first note that the probability density function ($p.d.f.$) for X is given by

$$f_X(x) = \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right) \cdot \Gamma\left(\frac{n}{2}\right)} \frac{m^{\frac{m}{2}} \cdot n^{\frac{n}{2}} \cdot x^{\frac{m}{2}-1}}{(n + m \cdot x)^{\frac{m+n}{2}}} \cdot 1_{\{x>0\}},$$

where $1_{\{A\}}$ is the indicator random variable, which takes the value of 1 if the event A occurs and takes the value of 0 otherwise. Then,

$$P(X \leq x) = K \int_0^x \frac{m^c \cdot n^d \cdot t^{c-1}}{(n + m \cdot t)^{c+d}} \, dt,$$

where $K = \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}$. Thus,

$$K \int_0^x \frac{m^c \cdot n^d \cdot t^{c-1}}{(n + m \cdot t)^{c+d}} \, dt = K \int_0^x \frac{m^c \cdot n^d \cdot t^{c-1}}{n^{c+d} \cdot \left(1 + \frac{m}{n} \cdot t\right)^{c+d}} \, dt.$$

Let $u = \frac{m}{n} \cdot t$, so that $du = \frac{m}{n} \cdot dt$. Hence, we have

$$K \int_0^x \frac{m^c \cdot n^d \cdot t^{c-1}}{n^{c+d} \cdot \left(1 + \frac{m}{n} \cdot t\right)^{c+d}} \, dt = K \int_0^{\frac{m}{n} \cdot x} \frac{m^c \cdot n^d \cdot \left(\frac{n}{m} \cdot u\right)^{c-1}}{n^{c+d} \cdot (1+u)^{c+d}} \cdot \frac{n}{m} \cdot du$$

$$= K \int_0^{\frac{m}{n} \cdot x} \frac{m^c \cdot n^d \cdot n^{c-1} \cdot n}{n^{c+d} \cdot m^{c-1} \cdot m} \cdot \frac{u^{c-1}}{(1+u)^{c+d}} \, du$$

$$= K \int_0^{\frac{m}{n} \cdot x} \frac{u^{c-1}}{(1+u)^{c+d}} \, du.$$

Let $z = \frac{u}{u+1}$, so that $u = \frac{z}{1-z}$. Thus,

$$du = \left(\frac{1}{1-z} + \frac{z}{(1-z)^2}\right) \cdot dz$$

$$= \frac{1}{(1-z)^2} \cdot dz.$$

Thus, we have

$$K \int_0^{\frac{m}{n} \cdot x} \frac{u^{c-1}}{(1+u)^{c+d}} \, du = K \int_0^{\frac{\frac{m}{n} \cdot x}{\frac{m}{n} \cdot x + 1}} \left(\frac{z}{1-z}\right)^{c-1} \cdot \frac{1}{\left(1 + \frac{z}{1-z}\right)^{c+d}} \cdot \frac{1}{(1-z)^2} \, dz$$

$$= K \int_0^{\frac{m \cdot x}{m \cdot x + n}} \frac{z^{c-1} \cdot (1-z)^{c+d}}{(1-z)^{c+1}} \, dz$$

$$= K \int_0^{\frac{m \cdot x}{m \cdot x + n}} z^{c-1} \cdot (1-z)^{d-1} \, dz$$

$$= P\left(Y \leq \frac{m \cdot x}{m \cdot x + n}\right).$$

Therefore, $P(X \leq x) = P\left(Y \leq \frac{m \cdot x}{m \cdot x + n}\right)$, as desired.

We should note that the Beta Distribution *p.d.f.* is defined for $z \in (0, 1)$. Here, $0 < \frac{m \cdot x}{m \cdot x + n} < 1$, so that our proposition holds.■

Proposition 12.2 : Let X and Y be the $R.V.$' s defined in Proposition 12.1. Let $K = \frac{\Gamma(c+d)}{\Gamma(c)\cdot\Gamma(d)}$,

where c, $d \geq 1$. Then,

(1) $P(X \leq x) = K \sum_{i=0}^{c-1} \binom{c-1}{i} (-1)^{c-(1+i)} \left(\frac{1-\left(\frac{n}{n+m\cdot x}\right)^{c+d-(i+1)}}{c+d-(i+1)} \right)$, if $c \in \mathbb{N}$,

(2) $P(X \leq x) = K \sum_{i=0}^{d-1} \binom{d-1}{i} (-1)^{d-(i+1)} \cdot \frac{\left(\frac{m\cdot x}{n+m\cdot x}\right)^{c+d-(i+1)}}{c+d-(i+1)}$, if $d \in \mathbb{N}$, and

(3) $P(X \leq x) = 2K \sum_{i=0}^{\frac{2c-1}{2}} \binom{\frac{2c-1}{2}}{i} (-1)^{\frac{2c-1}{2}-i} (P_{c+d-1-i} + Q_{c+d-1-i})$, if $c \cap d \notin \mathbb{N}$, where

$$P_k = \sum_{i=1}^{k} \frac{\text{Cos}^{2(k-i)+1}\,\theta \cdot \text{Sin}\,\theta \prod_{j=0}^{i-1}(2(k-j)+1)}{(2k+1)\prod_{j=0}^{i-1}(2(k-j))},$$

$$Q_k = \frac{\prod_{i=1}^{k}(2(k-i)+1)}{\prod_{i=0}^{k-1}(2(k-i))} \cdot \theta,$$

$k = c+d-1-i$, and $\theta = \text{Sin}^{-1}\sqrt{\frac{m\cdot x}{n+m\cdot x}}$. □

Proof :

(1) Recall, in Proposition 12.1 we showed that $P(X \leq x) = P\left(Y \leq \frac{m\cdot x}{m\cdot x+n}\right)$. Now,

$$P\left(Y \leq \frac{m\cdot x}{m\cdot x+n}\right) = K \int_0^{\frac{m\cdot x}{m\cdot x+n}} z^{c-1}(1-z)^{d-1}\,dz.$$

Let $u = 1-z$. Then $du = -dz$, so that

$$K \int_0^{\frac{m\cdot x}{m\cdot x+n}} z^{c-1}(1-z)^{d-1}\,dz = -K \int_1^{\frac{n}{m\cdot x+n}} (1-u)^{c-1} u^{d-1}\,du$$

$$= K \int_{\frac{n}{m\cdot x+n}}^{1} (1-u)^{c-1} u^{d-1}\,du.$$

The binomial expansion of the term $(1-u)^{c-1}$ is $\sum_{i=0}^{c-1} \binom{c-1}{i} (-1)^{c-(1+i)} (u)^{c-(1+i)}$, so that

$$K \int_{\frac{n}{m\cdot x+n}}^{1} (1-u)^{c-1} u^{d-1}\,du = K \int_{\frac{n}{m\cdot x+n}}^{1} \sum_{i=0}^{c-1} \binom{c-1}{i} (-1)^{c-(1+i)} (u)^{c-(1+i)} u^{d-1}\,du$$

$$= K \sum_{i=0}^{c-1} \binom{c-1}{i} (-1)^{c-(1+i)} \int_{\frac{n}{m\cdot x+n}}^{1} u^{c+d-1-i-1}\,du$$

$$= K \sum_{i=0}^{c-1} \binom{c-1}{i} (-1)^{c-(1+i)} \left(\frac{1 - \left(\frac{n}{m \cdot x + n}\right)^{c+d-(i+1)}}{c + d - (1 + i)} \right).$$

Thus, $P(X \leq x) = K \sum_{i=0}^{c-1} \binom{c-1}{i} (-1)^{c-(1+i)} \left(\frac{1 - \left(\frac{n}{m \cdot x + n}\right)^{c+d-(i+1)}}{c + d - (1 + i)} \right)$, as the proposition suggests.

(2) The binomial expansion for the term $(1 - z)^{d-1}$ of the integral $K \int_0^{\frac{m \cdot x}{m \cdot x + n}} z^{c-1} (1 - z)^{d-1} \, dz$ is

$$(1 - z)^{d-1} = \sum_{i=0}^{d-1} \binom{d-1}{i} (1)^i (-z)^{d-(1+i)}$$

$$= \sum_{i=0}^{d-1} \binom{d-1}{i} (-1)^{d-(1+i)} (z)^{d-(1+i)}.$$

Thus,

$$K \int_0^{\frac{m \cdot x}{m \cdot x + n}} z^{c-1} (1 - z)^{d-1} \, dz = K \int_0^{\frac{m \cdot x}{m \cdot x + n}} z^{c-1} \sum_{i=0}^{d-1} \binom{d-1}{i} (-1)^{d-(1+i)} (z)^{d-(1+i)} \, dz$$

$$= K \sum_{i=0}^{d-1} \binom{d-1}{i} (-1)^{d-(1+i)} \int_0^{\frac{m \cdot x}{m \cdot x + n}} z^{c-1+d-1-i} \, dz$$

$$= K \sum_{i=0}^{d-1} \binom{d-1}{i} (-1)^{d-(1+i)} \frac{\left(\frac{m \cdot x}{m \cdot x + n}\right)^{c+d-(i+1)}}{c + d - (i + 1)}.$$

Thus, if $d \in \mathbb{N}$, then $P\left(Y \leq \frac{m \cdot x}{m \cdot x + n}\right) = K \sum_{i=0}^{d-1} \binom{d-1}{i} (-1)^{d-(1+i)} \frac{\left(\frac{m \cdot x}{m \cdot x + n}\right)^{c+d-(i+1)}}{c + d - (i + 1)}$, as the proposition suggests.

(3) Again, recall in Proposition 12.1, we showed that $P(X \leq x) = P\left(Y \leq \frac{m \cdot x}{m \cdot x + n}\right)$. Now,

$$P\left(Y \leq \frac{m \cdot x}{m \cdot x + n}\right) = K \int_0^{\frac{m \cdot x}{m \cdot x + n}} z^{c-1} (1 - z)^{d-1} \, dz,$$

where c, $d \geq 1$, $c \cap d \in \mathbb{Q}$. Let $\frac{a}{2} = c - 1$, and let $\frac{b}{2} = d - 1$. This implies that $a = 2(c - 1)$ and $b = 2(d - 1)$.

Thus, Lemma 12.3 provides that

$$K \int_0^{\frac{m \cdot x}{m \cdot x + n}} z^{c-1} (1 - z)^{d-1} \, dz = 2K \int_0^{\text{Sin}^{-1} \sqrt{\frac{m \cdot x}{m \cdot x + n}}} \text{Sin}^{2c-1} \theta \cdot \text{Cos}^{2d-1} \theta \, d\theta$$

$$= 2K \sum_{i=0}^{\frac{2c-1}{2}} \binom{\frac{2c-1}{2}}{i} (-1)^{\frac{2c-1}{2} - i} (P_{c+d-1-i} + Q_{c+d-1-i}),$$

where

$$P_k = \sum_{i=1}^{k} \frac{\text{Cos}^{2(k-i)+1}\theta \cdot \text{Sin}\,\theta \prod_{j=0}^{i-1}(2(k-j)+1)}{(2k+1)\prod_{j=0}^{i-1}(2(k-j))},$$

$$Q_k = \frac{\prod_{i=1}^{k}(2(k-i)+1)}{\prod_{i=0}^{k-1}(2(k-i))} \cdot \theta,$$

$k = c + d - 1 - i$, and $\theta = \text{Sin}^{-1}\sqrt{\frac{m \cdot x}{n + m \cdot x}}$, as desired.

Therefore, if $c \cap d \notin \mathbb{N}$, then $P(X \leq x) = 2K \sum_{i=0}^{\frac{2c-1}{2}} \binom{\frac{2c-1}{2}}{i} (-1)^{\frac{2c-1}{2}} (P_{c+d-1-i} + Q_{c+d-1-i})$.■