THEOREM I (Permuted Ordering Algorithm): Let $n, m \in \mathbb{N}$ be arbitrary for which $n \geq m$; and let $N_0 \in \mathbb{N}$ be any positive integer satisfying the inequality

$$N_0 \leq \binom{n}{m}. \tag{1.1}$$

Consider the following algorithm:

<u>Step 1:</u> For $i < m$, let i be the number of "visits" to this step (first) of the algorithm. We define $k_i \in \mathbb{N}$ by

$$k_i = \min\left\{ t \in \mathbb{N} : N_{i-1} \leq \sum_{p=1}^{t} \binom{n-p-j_{i-1}}{m-i} \right\}, \tag{1.2}$$

where $j_0 = 0$. Assign the updated value, j_i, by $j_i = k_i + j_{i-1}$.

<u>Step 2:</u> As with the first step of the algorithm, for $i < m$, let i be the number of visits to this step (second) of the algorithm. Assign the updated value, N_i, by

$$N_i = N_{i-1} - I(j_i > j_{i-1} + 1) \cdot \sum_{t=1}^{k_i - 1} \binom{n-t-j_{i-1}}{m-i}, \tag{1.3}$$

where $I(A)$ is the indicator random variable for the event A, whose value is given by

$$I(A) = \begin{cases} 1, & \text{if } A \text{ occurs} \\ 0, & \text{otherwise.} \end{cases}$$

<u>Step 3:</u> Let i be the number of visits to this step (third) of the algorithm. If $i < m - 1$, then proceed to Step 1 of the algorithm. Otherwise, assign the updated value, j_m, by $j_m = j_{m-1} + N_{m-1}$. The algorithm is complete.□

For the values n, m, and N_0 given, the following four conditions hold for the above algorithm:

(a) There exists a sequence, $\{j_i\}_{i=1}^{m}$, $j_i \in \mathbb{N}$, such that $j_1 < j_2 < \cdots < j_m$;

(b) The sequence $\{N_i\}_{i=1}^{m-1}$ is non-increasing, such that $N_i > 0$ for all $i \leq m - 1$;

(c) $j_m \leq n$; and

(d) For fixed values of n and m, the sequence given by (a) is unique to the value of N_0.

Proof: (a) (Existence) - From the definition of k_i, given by (1.2), it follows that $k_i \geq 1$ for all $i = 1, 2, ..., m-1$. Thus, the updated value for j_i, is such that

$$j_i = k_i + j_{i-1} \geq 1 + j_{i-1} > j_{i-1} \in \mathbb{N}, \quad \forall i = 1, 2, ..., m-1.$$

Thus, to show (a), it is left to show that $j_m > j_{m-1}$. To show this, we use a proof by contradiction. Indeed, suppose that $j_m \leq j_{m-1}$. Then, it must be true that $N_{m-1} = j_m - j_{m-1} \leq 0$. Now, it is given that $N_0 \geq 1$. Hence, there exists $\delta \in \mathbb{N}$, $\delta \leq m-1$, such that $\delta = \min\{p \in \mathbb{N} : N_p \leq 0\}$. This implies that $N_{\delta-1} > 0$, where $\delta - 1 \leq m-2$. Moreover, by inspection, we see that a necessary and sufficient condition for $N_\delta < 0$, is that this "sign change" occurs during the δ^{th} visit to Step 1 and Step 2 of the algorithm. Now, from (1.2), utilizing the fact that (1.3) provides that $N_i \neq N_{i-1}$ only when $k_i > 1$, it holds

$$\sum_{t=1}^{k_\delta - 1} \binom{n - t - j_{\delta-1}}{m - \delta} < N_{\delta-1} \leq \sum_{t=1}^{k_\delta} \binom{n - t - j_{\delta-1}}{m - \delta}, \tag{1.4}$$

where $k_\delta > 1$. Hence, (1.3) and (1.4) imply

$$N_\delta = N_{\delta-1} - I\left(j_\delta > j_{\delta-1} + 1\right) \cdot \sum_{t=1}^{k_\delta - 1} \binom{n - t - j_{\delta-1}}{m - \delta} > 0.$$

Hence, $N_\delta \leq 0$ and $N_\delta > 0$, a contradiction. This implies that there does not exist a $\delta \in \mathbb{N}$, $\delta \leq m-1$ for which $\delta = \min\{p \in \mathbb{N} : N_p \leq 0\}$. Therefore, $N_i > 0$ for all $i \leq m-1$. In particular, $N_{m-1} > 0$, which provides the desired result,

$$0 < N_{m-1} = j_m - j_{m-1} \qquad \Longleftrightarrow \qquad j_{m-1} < j_m.$$

Therefore, there exists a sequence, $\{j_i\}_{i=1}^m$, $j_i \in \mathbb{N}$, such that $j_1 < j_2 < \cdots < j_m$.$\square$

(b) We have already established that $N_i > 0$ for all $i \leq m-1$. Thus, it is left to show that the N_i are non-increasing. To show that the N_i are non-increasing, we use a proof by contradiction. Indeed, suppose there exists $i \leq m-1$ for which $N_i > N_{i-1}$. Let $\delta = \min\{p \in \mathbb{N} : N_p > N_{p-1}\}$. As we have elucidated previously, from (1.2) and (1.3), in order for $N_\delta \neq N_{\delta-1}$, it must hold that $k_\delta > 1$. Moreover, since $N_{\delta-1} > 0$, in order for (1.3) and the relation $N_\delta > N_{\delta-1}$ to simultaneously hold, by the definition of δ, it must be true that

$$\sum_{t=1}^{k_\delta - 1} \binom{n - t - j_{\delta-1}}{m - \delta} < 0 \qquad \text{and} \qquad 0 < N_{\delta-1} \leq \sum_{t=1}^{k_\delta} \binom{n - t - j_{\delta-1}}{m - \delta}. \tag{1.5}$$

But, in order for (1.5) to hold: (i) the value of $m-\delta$ must be and odd integer; and (ii) there exists $t' < k_\delta$, for which

$$\binom{n-k_\delta+1-j_{\delta-1}}{m-\delta} < \ldots < \binom{n-t'-j_{\delta-1}}{m-\delta} < 0 \le \binom{n-t'+1-j_{\delta-1}}{m-\delta} < \ldots < \binom{n-1-j_{\delta-1}}{m-\delta}.$$

Moreover, it holds

$$\binom{n-k_\delta-j_{\delta-1}}{m-\delta} < \binom{n-k_\delta+1-j_{\delta-1}}{m-\delta} < 0 \qquad \Longleftrightarrow \qquad \sum_{t=1}^{k_\delta} \binom{n-t-j_{\delta-1}}{m-\delta} < 0,$$

a contradiction. Therefore, the sequence $\{N_i\}_{i=1}^{m-1}$ is non-increasing, and $N_i > 0$ for all $i \le m-1$.$\square$

(c) We need to show that $j_m \le n$. To show this result, we again use a proof by contradiction. Indeed, suppose $j_m > n$. Then, there exists $\delta \in \mathbb{N}$ for which $j_m = n+\delta$. Here, we note from Step 3 of the algorithm, it follows that $j_m = j_{m-1} + N_{m-1} = n+\delta$. This implies that

$$j_{m-1} = n+\delta-N_{m-1} \qquad \text{and} \qquad N_{m-1} = n+\delta-j_{m-1}. \tag{1.6}$$

Clearly, $k_{m-1} \ge 1$. First, suppose that $k_{m-1} = 1$. Then, from Steps 1 and 2 of the algorithm, it follows that $N_{m-2} = N_{m-1}$. From Step 1 of the algorithm, it is,

$$\begin{aligned} N_{m-1} &= N_{m-2} \\ &\le \sum_{t=1}^{k_{m-1}} \binom{n-t-j_{m-2}}{m-(m-1)} \\ &= \binom{n-k_{m-1}-j_{m-2}}{1} \\ &= n-j_{m-1}, \end{aligned}$$

of which (1.6) provides that $n+\delta-j_{m-1} \le n-j_{m-1}$. But, this holds iff $\delta \le 0$. This is a contradiction. Hence, it must be that $k_{m-1} > 1$. Hence, from (1.3), we have

$$\begin{aligned} \sum_{t=1}^{k_{m-1}-1} \binom{n-t-j_{m-2}}{m-(m-1)} + N_{m-1} = N_{m-2} \\ \le \sum_{t=1}^{k_{m-1}} \binom{n-t-j_{m-2}}{m-(m-1)}, \end{aligned}$$

so that

$$N_{m-1} \overset{(1.6)}{=} n+\delta-j_{m-1} \le n-k_{m-1}-j_{m-2} = n-j_{m-1} \qquad \Longleftrightarrow \qquad \delta \le 0,$$

a contradiction. Thus, it must be true that our original supposition that $j_m > n$ is false. Therefore, $j_m \leq n$, as desired.$\square$

(d) (uniqueness) - Let the sequence $\{j_i\}_{i=1}^{m}$, for which $j_1 < j_2 < \cdots < j_m$, be arbitrary. It is,

$$\begin{aligned}(N_0 - N_1) + (N_1 - N_2) + \ldots + (N_{m-2} - N_{m-1}) &\overset{(A)}{=} \sum_{i=1}^{m-1} I(j_i > j_{i-1} + 1) \sum_{t=1}^{k_i - 1} \binom{n - t - j_{i-1}}{m - i} \\ &= N_0 - N_{m-1} \\ &= N_0 - (j_m - j_{m-1}), \end{aligned} \tag{1.7}$$

where we note that the terms of the summand for (A) are uniquely determined by the sequence $\{j_i\}_{i=1}^{m-1}$. Moreover, the value of N_0 is uniquely determined by the sequence $\{j_i\}_{i=1}^{m}$, since (1.7) is completely determined by $\{j_i\}_{i=1}^{m}$ and the value of N_0. Therefore, for fixed values of n and m, the sequence $\{j_i\}_{i=1}^{m}$, is unique for the value of N_0. $\blacksquare$

Example of Theorem 1: Given, $n = 10$, $m = 4$, and $N_0 = 85$, let's derive the sequence $\{j_i\}_{i=1}^{m}$. For the first visit to Step 1, by inspection with $p = 1$, we find $\binom{n-1}{m-i} = \binom{9}{3} = 84 < N_0$. Also, when $t = 2$, we find,

$$N_0 \leq \sum_{p=1}^{t} \binom{n-p-j_{i-1}}{m-i} = \binom{9}{3} + \binom{8}{3} = 140,$$

so that $k_1 = 2$ and $j_1 = 2$. Hence, from Step 2, it is

$$\begin{aligned} N_1 &= N_0 - I(j_1 > j_0 + 1) \cdot \sum_{t=1}^{k_1 - 1} \binom{n-t-j_0}{m-1} \\ &= 85 - \binom{10-1}{4-1} \\ &= 1. \end{aligned}$$

Table 1 below, summarizes the results for this example, where we find that $\{j_i\}_{i=1}^{m} = \{2, 3, 4, 5\}$. Also, Table 2 below, summarizes the results for $n = 10$, $m = 4$, and $N_0 = 100$.

Table 1:

m	0	1	2	3	4
N_m	85	1	1	1	–
k_m	–	2	1	1	–
j_m	0	2	3	4	5

Table 2:

m	0	1	2	3	4
N_m	100	16	16	1	–
k_m	–	2	1	4	–
j_m	0	2	3	7	8

■